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JHEP07(2009)078

(http://iopscience.iop.org/1126-6708/2009/07/078)

Published by IOP Publishing for SISSA

Received: *June 18, 2009*
Accepted: *July 6, 2009*
Published: *July 21, 2009*

Geometry of the $N = 2$ supersymmetric sigma model with Euclidean worldsheet

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Abstract: We investigate the target space geometry of supersymmetric sigma models in two dimensions with Euclidean signature, and the conditions for $N = 2$ supersymmetry. For a real action, the geometry for the $N = 2$ model is not the generalized Kähler geometry that arises for Lorentzian signature, but is an interesting modification of this which is not a complex geometry.

Keywords: Extended Supersymmetry, Differential and Algebraic Geometry, Sigma Models

ArXiv ePrint: 0906.2741

doi:10.1088/1126-6708/2009/07/078

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1 Introduction

In this paper we discuss $N = 2$ supersymmetric sigma models in 2 dimensions with Euclidean signature. One such model arises when the usual Lorentzian signature $N = 1$ model is Wick-rotated and then required to have additional non-manifest supersymmetries. In this case, the Wess-Zumino term is imaginary and the action complex. This model was studied in connection with topological theories in [5]. Below we briefly discuss the target space geometry in this case. The R-symmetry group is $\mathrm{SO}(2) \times \mathrm{SO}(1,1)$ [4, 5] allowing an A-twist in which the SO(2) factor is twisted with the 2d rotation group SO(2) but not a B-twist. In [5] we considered the complexification of this model with R-symmetry $\mathrm{SO}(2,\mathbb{C}) \times \mathrm{SO}(2,\mathbb{C})$ allowing both an A-twist and a B-twist with the complexified Lorentz group, which is also $\mathrm{SO}(2,\mathbb{C})$.

The main result of the paper concerns the Euclidean model with real action and real WZ term. The analysis closely follows that of GHR, (Gates, Hull and Roček) [1] in the Lorentzian case, i.e., we make an ansatz for the extra supersymmetries and find the constraints on the target space geometry that follow from closure of the algebra and invariance of the action. We find a curious generalization of complex geometry, which has a complex tensor J that satisfies $J^2 = -1$ and has vanishing Nijenhuis tensor. By complex tensor, we mean that it has components in a real coordinate system that are complex, whereas for a complex structure, the components would be real. We briefly discuss the underlying geometry.

We give the $N = 2$ superspace formulation for the case in which the supersymmetry algebra closes off-shell. In this case, the target space geometry has a metric of indefinite signature and two Yano f-structures [9].

2 Sigma models

The two-dimensional nonlinear sigma model has the action

$$S = -\frac{1}{4}\int_{\Sigma} d^2\sigma\,\sqrt{h}\,[h^{\mu\nu}\partial_\mu\phi^i\partial_\nu\phi^j g_{ij}(\phi) + \epsilon^{\mu\nu}\partial_\mu\phi^i\partial_\nu\phi^j B_{ij}(\phi)]\,, \tag{2.1}$$

for maps $\{\phi\}$ from a two dimensional manifold Σ to a d-dimensional target space M:

$$\phi:\ \Sigma \to M\ . \tag{2.2}$$

specified locally by functions $\phi^i(\sigma)$ giving the dependence of the real coordinates ϕ^i of M on the real coordinates σ^μ of Σ. The target manifold M has a metric g and 2-form potential B, while Σ has a metric $h_{\mu\nu}$ with $h = |\det(h_{\mu\nu})|$. The potential B need only be locally defined, but there is a globally-defined closed 3-form field strength H such that locally $H = dB$. The equations of motion depend on B only through the 3-form field strength H and so are well-defined.

In the usual case, the metric $h_{\mu\nu}$ has Lorentzian signature and $g_{ij}(\phi)$ and $B_{ij}(\phi)$ have real components. The Euclidean version of this used in the path integral (given by a Wick rotation in the case in which $h_{\mu\nu}$ is flat) is

$$S = -\frac{1}{4}\int_{\Sigma} d^2\sigma\,\sqrt{h}\,[h^{\mu\nu}\partial_\mu\phi^i\partial_\nu\phi^j g_{ij}(\phi) + i\epsilon^{\mu\nu}\partial_\mu\phi^i\partial_\nu\phi^j B_{ij}(\phi)]\,, \tag{2.3}$$

with $h_{\mu\nu}$ a Euclidean signature metric. Note that the term involving B is now pure imaginary, so that the action is complex. For both the Lorentzian and Wick-rotated case, the quantum theory is well-defined if H is a globally-defined 3-form that represents an integral cohomology class, $H \in H^3(\mathbb{Z})$. Geometrically this means that there is a gerbe with curvature H and connection B_α in each coordinate patch $\mathcal{O}_\alpha$. For the path integral, if $H_2(M)$ is non-trivial, it is not sufficient to specify H , and a choice of B must be made. Then the term containing the B-field

$$e^{2\pi i\int \phi^*(B)} \tag{2.4}$$

defines the holonomy of a gerbe over the embedding of the world sheet. For further details on gerbes and gerbe holonomy see [3, 6].

For Euclidean signature one can also consider the real action (2.1) with $h_{\mu\nu}$ a Euclidean signature metric. For the action to be well-defined, B should be a globally-defined 2-form. However, the field equations are well-defined provided only that H is a well-defined 3-form, so that a classical theory exists for any closed 3-form H.

This paper will investigate the $N = 2$ supersymmetrisations of both the real action (2.1) and the complex action (2.3) for Euclidean $h_{\mu\nu}$. The motivation for this comes from our investigation of topological twistings [5], where both cases played a role.

An $N = 1$ supersymmetric version of these sigma model are obtained by promoting the ϕ's to $N = 1$ superfields $\Phi(\sigma, \theta)$ depending on fermionic coordinates $\theta^\pm$, where θ^+ has positive chirality and θ^- has negative chirality. In Lorentzian signature, $\theta^\pm$ are independent real Majorana-Weyl spinors, while in Euclidean signature they are complex conjugate Weyl spinors, $(\theta^+)^* = \theta^-$. The corresponding supercovariant spinor derivatives are $D_\pm$; see [4, 5] for further discussion of our conventions.

For both the Lorentzian sigma model with action (2.1) and Euclidean sigma model with complex action (2.3), the supersymmetric action is (taking h to be flat)

$$S = -\frac{1}{4}\int d^2\sigma d^2\theta \,(D_+\Phi^i E_{ij}(\Phi) D_-\Phi^j)\,, \tag{2.5}$$

where

$$E_{ij} = g_{ij} + B_{ij}\,, \tag{2.6}$$

By contrast, for the Euclidean sigma model with real action (2.1), the $N = 1$ supersymmetric version is again given by (2.5) but now

$$E_{ij} = g_{ij} + iB_{ij}\,, \tag{2.7}$$

is complex.

For special target space geometries, these $N = 1$ sigma models can have extra supersymmetries. For example, the Lorentzian sigma model will have $N = 2$ supersymmetry provided the target space has the bihermitean geometry of GHR [1], which has recently been given a new formulation in terms of Generalized Kähler geometry [2, 7]. Here we will examine the geometries needed for the real and Wick-rotated $N = 1$ Euclidean sigma models to have $N = 2$ supersymmetry.

3 Geometry of the classical models

In this section we briefly review the geometric structure of the target spaces for the Lorentzian and Wick-rotated models. We then present our main results that concern the geometry for the Euclidean model with real action.

3.1 The Lorentzian $N = 2$ model

We start with the Lorentzian signature $N = 1$ supersymmetric action (2.5) with (2.6) and follow the analysis of [1]. The general ansatz for an extra right and left supersymmetry is

$$\delta_\epsilon \Phi^i = iJ_+{}^i{}_j(\epsilon_- D_+\Phi^j) + iJ_-{}^i{}_j(\epsilon_+ D_-\Phi^j)\,, \tag{3.1}$$

where $\epsilon_\pm$ are independent real supersymmetry transformation parameters and $J_\pm$ are some mixed real tensors on M. Closure of the supersymmetry algebra and invariance of the action then impose conditions on $J_\pm$. Closure requires that $J_\pm$ are complex structures,

$$J_\pm^2 = -1\,, \quad \mathcal{N}(J_\pm) = 0\,, \tag{3.2}$$

where $\mathcal{N}(J)$ denotes the Nijenhuis tensor. Invariance of the action requires that they are also covariantly constant with respect to connections with torsion,

$$\nabla^\pm J_\pm = 0\,, \tag{3.3}$$

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and the metric g is hermitean with respect to both

$$J^t_\pm g J_\pm = g \ . \tag{3.4}$$

The connections with torsion are constructed from the Levi-Civita connection Γ and the 3-form $H = dB$:

$$\Gamma^\pm = \Gamma \pm \frac{1}{2} g^{-1} H \ . \tag{3.5}$$

Then M has a GHR bihermitian geometry [1].

3.2 The Wick rotated $N = 2$ model

Consider next the 'Wick-rotated' model given by $N = 1$ supersymmetric action (2.5) with (2.6) and Euclidean world-sheet metric, so that the component expansion has bosonic part (2.3) with imaginary WZ term. The anasatz for the extra supersymmetry is again (3.1) but now all spinors are complex, with

$$(\epsilon_\pm)^* = \epsilon_\mp \,, \quad (D_\pm)^* = D_\mp \ . \tag{3.6}$$

The algebra of the supercovariant derivatives is

$$\begin{aligned} \{D_+, D_+\} &= \partial \,, \\ \{D_-, D_-\} &= \bar{\partial} \,, \end{aligned} \tag{3.7}$$

where the partial derivatives on the right are derivatives with respect to $z = \sigma^1 + i\sigma^2$ and $\bar{z} = \sigma^1 - i\sigma^2$ respectively. Closure of the algebra and invariance of the action give the same set of equations (3.2)–(3.5) as for the Lorentzian case. However, the reality conditions on Φ and the transformations (3.1) give us the condition

$$J^*_+ = J_- \ . \tag{3.8}$$

The complex conjugate of (3.3) now yields

$$\nabla^\pm J_\mp = 0 \,, \tag{3.9}$$

which together with (3.3) implies the Kähler equation,

$$\nabla J_\pm = 0 \,, \tag{3.10}$$

and

$$H = 0 \ . \tag{3.11}$$

Indeed this should not come as surprise. The Wick-rotated action is complex and so the real and imaginary parts must be separately invariant, so that the geometry must be Kähler and the WZ term trivial. Then B is a connection on a flat gerbe and the expression $e^{2\pi i \int \phi^*(B)}$ is well-defined and gives us the holonomy of the flat gerbe.

3.3 The Euclidean $N = 2$ model with real action

We now consider the action (2.5) with Euclidean world-sheet and

$$E_{ij} = g_{ij} + iB_{ij} \,, \tag{3.12}$$

so that the component expansion has the bosonic term (2.1) with real B-field term. The anasatz for the extra supersymmetry is again (3.1) but now all spinors are complex, with the reality conditions on Φ and the transformations (3.1) again giving $J_+^* = J_-$. Then the second supersymmetry variation (3.1) becomes

$$\delta_\epsilon \Phi^i = iJ^i{}_j(\epsilon_- D_+ \Phi^j) + iJ^{*i}{}_j(\epsilon_+ D_- \Phi^j) \,, \tag{3.13}$$

where J^* is the complex conjugate to $J = J_+$. Alternatively we can split J into real and imaginary parts

$$J = f + i\tilde{f} \,, \tag{3.14}$$

where f and $\tilde{f}$ are real tensors, so that the transformation (3.13) becomes

$$\delta_\epsilon \Phi^i = if^i{}_j(\epsilon_- D_+ + \epsilon_+ D_-)\Phi^j + \tilde{f}^i{}_j(\epsilon_+ D_- - \epsilon_- D_+)\Phi^j \,. \tag{3.15}$$

The conditions for supersymmetry following [1] are similar to before, but with extra factors of i. The on-shell closure of the supersymmetry algebra implies that

$$J^2 = -1$$

and its Nijenhuis tensor vanishes, $\mathcal{N}(J) = 0$. Invariance of the action under the second supersymmetry (3.1) requires the metric g must satisfy

$$J^t g J = g \tag{3.16}$$

together with

$$\nabla_H J = 0 \,, \tag{3.17}$$

where ∇_H has connection

$$\Gamma_H = \Gamma + \frac{i}{2} g^{-1} H \,.$$

Thus formally our new conditions are similar to the generalized Kähler geometry, but now $J_+ = J$, $J_- = J^*$ are complex tensors and the torsion term in the connection now has a factor of i. Thus the conditions are formally similar to those for generalized Kähler geometry, but the different reality properties and extra factors of i means that the implications of these conditions will be quite different.

The target manifold M is not a complex manifold in the standard sense. We still can define the projectors

$$p_\pm = \frac{1}{2}(1 \pm iJ) \,, \qquad p_\pm^* = \frac{1}{2}(1 \mp iJ^*) \,, \tag{3.18}$$

which would give us four integrable complex distributions on the complexified tangent bundle $TM_{\mathbb{C}}$. However we would not be able to define the decomposition of a vector into holomorphic and antiholomorphic parts. For example, the projector p_- would define the "holomorphic" vectors $T^{(1,0)}M$, but unlike the complex manifold we have now

$$T^{(1,0)}M \cap T^{(0,1)}M \neq \emptyset\,, \tag{3.19}$$

where $T^{(0,1)}M$ is the subbundle complex conjugate to $T^{(1,0)}M$.

Using the real and imaginary parts of J introduced in (3.14) the condition $J^2 = -1$ becomes

$$\begin{aligned} f^2 - \tilde{f}^2 &= -1\,, \\ \{f, \tilde{f}\} &= 0\ . \end{aligned} \tag{3.20}$$

In terms of real tensors, the condition (3.17) can be written as two real equations,

$$\begin{aligned} \nabla f &= \frac{1}{2} g^{-1} H \tilde{f}\,, \\ \nabla \tilde{f} &= -\frac{1}{2} g^{-1} H f\ . \end{aligned} \tag{3.21}$$

Furthermore, as in the generalized Kähler case [8] we can define two real Poisson structures

$$\pi_+ = \frac{1}{2}(J + J^*) g^{-1} = f g^{-1}\,, \tag{3.22}$$

$$\pi_- = \frac{1}{2i}(J - J^*) g^{-1} = \tilde{f} g^{-1}\,, \tag{3.23}$$

which define symplectic foliations. Locally we can choose the coordinates adapted to these foliations and f, $\tilde{f}$ look relatively simple in those coordinates.

3.4 Off-shell closure and f-structures

The $N = 2$ superalgebra will close off-shell only if J and J^* commute.[1] In this case the condition $[J, J^*] = 0$ becomes

$$f \tilde{f} = 0\ . \tag{3.24}$$

Then at least one of the two structures $f, \tilde{f}$ must be degenerate. Then f and $\tilde{f}$ satisfy

$$f^3 + f = 0\,, \tag{3.25}$$

$$\tilde{f}^3 - \tilde{f} = 0\ . \tag{3.26}$$

Equation (3.25) is the generalization of an almost complex structure condition ($f^2 = -1$) to allow the possibility of f being a degenerate tensor. A tensor f of constant rank satisfying (3.25) is a *Yano f-structure* [9]. Similarly, equation (3.26) is the generalization of an almost product structure ($\tilde{f}^2 = 1$) condition with $\tilde{f}$ being possibly degenerate and gives a generalised f-structure.

[1] If auxiliary fields are included, the situation changes. See [7].

Let

$$P = -f^2 \ . \tag{3.27}$$

Then

$$P^2 = P \,, \tag{3.28}$$

so that P is a projector. At a point, if the rank of P is r, then we can choose a basis in which P has a block form

$$P = \left[\begin{array}{c|c} \mathbf{0} & \\ \hline & 1\!\!1 \end{array}\right] , \tag{3.29}$$

where $1\!\!1$ is the $r \times r$ unit matrix and $\mathbf{0}$ is the $(D-r) \times (D-r)$ zero matrix. Then

$$f = \left[\begin{array}{c|c} \mathbf{0} & \\ \hline & j \end{array}\right] , \tag{3.30}$$

where j is an $r \times r$ non -degenerate matrix satisfying

$$j^2 = -1\!\!1 \ .$$

This implies that r is even, $r = 2q$, and one can choose a basis so that

$$j = \begin{pmatrix} 0 & 1\!\!1 \\ -1\!\!1 & 0 \end{pmatrix} \ . \tag{3.31}$$

Next, since $f\tilde{f} = 0$, $\tilde{f}$ has the block form

$$\tilde{f} = \left[\begin{array}{c|c} \pi & \\ \hline & \mathbf{0} \end{array}\right] , \tag{3.32}$$

where π is a $(D-r) \times (D-r)$ matrix satisfying

$$\pi^3 - \pi = 0 \ .$$

Then π has eigenvalues $\pm 1, 0$ and take the form

$$\begin{pmatrix} 1\!\!1 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -1\!\!1 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix} \tag{3.33}$$

split into blocks of dimension a, b, c with $a + b + c + r = D$. If the number c of zero eigenvalues is non-zero, then there will be a subspace on which the 2nd supersymmetry does not act. If it is non-degenerate, then

$$\pi^2 = 1\!\!1$$

and we will mostly be interested in this non-degenerate case.

Finally

$$J = \left[\begin{array}{c|c} i\pi & 0 \\ \hline 0 & j \end{array}\right] \ . \tag{3.34}$$

The vanishing of the Nijenhuis tensor for J implies that we can choose coordinates so that J takes this form on a patch.

As in Lorentzian signature [1], we can (for non-degenerate π) define a local *product structure* defined by a real tensor Π,

$$\Pi = JJ^* \tag{3.35}$$

satisfying

$$\Pi^2 = \mathbb{1}\,,$$

which takes the form

$$\Pi = \left[\begin{array}{c|c} \mathbb{1} & 0 \\ \hline 0 & -\mathbb{1} \end{array}\right] \tag{3.36}$$

and this local product structure Π is integrable.

We will see in the next section that $N = 2$ superspace naturally gives geometries with $a = b = p, c = 0$ and which admit a local description in terms of single real function, very much in analogy with the Kähler and generalized Kähler cases.

3.5 The Euclidean $N = 2$ model with real action in $N = 2$ superspace

The $N = 1$-supersymmetric action (2.5) with an extra supersymmetry (3.1) that closes off-shell[2] can be reformulated in Euclidean $N = 2$ superspace:

$$S = 2\int d^2z\, d^2\theta\, d^2\bar\theta\; K(\phi, \bar\phi, \chi, \tilde\chi)\ . \tag{3.37}$$

We remind the reader that in Euclidean signature, the conjugation relations for $N = 2$ spinor derivatives are

$$(\mathbb{D}_\pm)^\dagger = \bar{\mathbb{D}}_\mp\ . \tag{3.38}$$

In (3.37) $\phi^u \quad u = 1, ..., p$ are chiral superfields ($\bar{\mathbb{D}}_+\phi = \bar{\mathbb{D}}_-\phi = 0$) and $\bar\phi^u$ their complex conjugates ($\mathbb{D}_+\bar\phi = \mathbb{D}_-\bar\phi = 0$). The fields χ^a are mixed chiral fields satisfying

$$\bar{\mathbb{D}}_+\chi = \mathbb{D}_-\chi = 0$$

where $a = 1, ..., q$. In Euclidean signature its complex conjugated field $\bar\chi$ is still a mixed chiral field, as

$$\bar{\mathbb{D}}_+\bar\chi = \mathbb{D}_-\bar\chi = 0 \tag{3.39}$$

so there is no loss of generality in taking the fields χ^a to be real. The fields $\tilde\chi^a$ are real mixed anti-chiral fields satisfying

$$\mathbb{D}_+\tilde\chi = \bar{\mathbb{D}}_-\tilde\chi = 0\ . \tag{3.40}$$

[2] In Lorentzian signature, a complete description covering all off-shell cases requires additional $N = (2, 2)$ semi-chiral fields.

We write the action in terms of q real mixed chiral fields χ^a and an equal number[3] of real mixed anti-chiral fields $\tilde{\chi}^a$. This structure of the potential K was first introduced and discussed in [4].

We now relate this to the action in section 3.3. Note that the bosonic part of the action is (2.1), which can be written using complex world-sheet coordinates $z, \bar{z}$ as

$$S = -\int_\Sigma d^2z(g_{ij}\partial\phi^i\bar{\partial}\phi^j - iB_{ij}\partial\phi^i\bar{\partial}\phi^j)\ . \tag{3.41}$$

We write the action (3.37) as

$$S = \int d^2z(\mathbb{D}_-\bar{\mathbb{D}}_-\mathbb{D}_+\bar{\mathbb{D}}_+ + \bar{\mathbb{D}}_-\mathbb{D}_-\bar{\mathbb{D}}_+\mathbb{D}_+)K(\phi,\bar{\phi},\chi,\tilde{\chi})\Big|\,, \tag{3.42}$$

where $(..)\Big|$ denotes taking the $\theta = 0$ part. The bosonic part of the action then becomes

$$\begin{aligned} S = \int d^2z\Big(&-K_{,\bar{u}v}\,\partial\phi^v\bar{\partial}\bar{\phi}^u - K_{,u\bar{v}}\,\partial\bar{\phi}^v\bar{\partial}\phi^u \\ &+K_{,\tilde{a}b}\,\partial\chi^b\bar{\partial}\tilde{\chi}^a + K_{,a\tilde{b}}\,\partial\tilde{\chi}^b\bar{\partial}\chi^a \\ &+K_{,u\bar{v}}\,\partial\bar{\phi}^v\bar{\partial}\phi^u - K_{,\bar{u}v}\,\partial\phi^v\bar{\partial}\bar{\phi}^u \\ &+K_{,ua}\,\partial\chi^a\bar{\partial}\phi^u - K_{,au}\,\partial\phi^u\bar{\partial}\chi^a \\ &+K_{,a\bar{u}}\,\partial\bar{\phi}^u\bar{\partial}\chi^a - K_{,\bar{u}a}\,\partial\chi^a\bar{\partial}\bar{\phi}^u\Big)\ . \end{aligned} \tag{3.43}$$

Comparing the actions (3.41) and (3.43), we learn about the geometry of the target space manifold. The metric g has a block diagonal structure,

$$\mathbf{g} = \left[\begin{array}{cc|cc} 0 & K_{,u\bar{v}} & & \\ K_{,\bar{u}v} & 0 & & \\ \hline & & 0 & -K_{,a\tilde{b}} \\ & & -K_{,\tilde{a}b} & 0 \end{array}\right]\,, \tag{3.44}$$

where we have a block with $2p \times 2p$ entries for the chiral sector and a block of $2q \times 2q$ for the mixed chiral sector. The chiral sector block of dimension $2p$ has Euclidean signature while the mixed chiral sector block of dimension $2q$ has a metric of split signature (q, q) with q positive eigenvalues and q negative ones.

The 2-form B has off-diagonal blocks mixing chiral with mixed chiral derivatives plus an extra bloc for the chiral sector,

$$\mathbf{B} = \left[\begin{array}{cc|cc} 0 & -iK_{,u\bar{v}} & -iK_{,ua} & 0 \\ iK_{,\bar{u}v} & 0 & iK_{,\bar{u}a} & 0 \\ \hline iK_{,au} & -iK_{,a\bar{u}} & & \\ 0 & 0 & & \end{array}\right]\ . \tag{3.45}$$

It has a different form from that in the standard GHR-gauge [1], which gives a B-field that is complex in Euclidean signature. Here we use an alternative gauge in which B is real when written in real coordinates.

[3] A different number of mixed chiral and mixed anti-chiral fields leads to degenerate models.

We now turn to the structures J and J^* that appear in the supersymmetry transformations (3.13), following [1]. The $N = 2$ superspace formulation makes the extra supersymmetry manifest. Expanding into $N = 1$ superfields gives transformations of the form (3.13) and from these one can read off the structures J and J^*, which are constant in this coordinate system. We define the Weyl $N = 1$ spinor derivative $D_\pm$, and the generator of the non-manifest supersymmetry $Q_\pm$,

$$\begin{aligned} D_\pm &= \frac{1}{\sqrt{2}}(\mathbb{D}_\pm + \bar{\mathbb{D}}_\pm)\,, \\ Q_\pm &= \frac{i}{\sqrt{2}}(\mathbb{D}_\pm - \bar{\mathbb{D}}_\pm)\;. \end{aligned} \tag{3.46}$$

The $N = 1$ algebra with the property (3.6) and $\{D_\pm, Q_\pm\} = 0$ follow from the $N = 2$ algebra and the property (3.38). The Q-transformations of the $N = 1$ fields $(\phi,\, \bar{\phi},\, \chi,\, \tilde{\chi})$ are

$$\begin{aligned} \delta_\epsilon \phi^u &= i\epsilon_- Q_+ \phi^u + i\epsilon_+ Q_- \phi^u \\ &= -\epsilon_- D_+ \phi^u - \epsilon_+ D_- \phi^u\,, \\ \delta_\epsilon \bar{\phi}^u &= \epsilon_- D_+ \bar{\phi}^u + \epsilon_+ D_- \bar{\phi}^u\,, \\ \delta_\epsilon \chi^a &= -\epsilon_- D_+ \chi^a + \epsilon_+ D_- \chi^a\,, \\ \delta_\epsilon \tilde{\chi}^a &= \epsilon_- D_+ \tilde{\chi}^a - \epsilon_+ D_- \tilde{\chi}^a\;. \end{aligned} \tag{3.47}$$

To relate this to the structure that we found for J in the previous section we need to expand the real $N = 1$ superfield Φ in real components, so we need to split the $N = 2$ chiral superfield ϕ and its antichiral partner $\bar{\phi}$ into their real components,

$$\phi = \phi_1 + i\phi_2\,,$$

$$\bar{\phi} = \phi_1 - i\phi_2\;.$$

Writing the $N = 1$ superfields Φ in terms of the real $N = 1$ superfields $(\chi,\, \tilde{\chi},\, \phi_1,\, \phi_2)$, we can read off the J in transformation (3.13) to be

$$J = \left[\begin{array}{cc|cc} i & 0 & & \\ 0 & -i & & \\ \hline & & 0 & -1 \\ & & 1 & 0 \end{array}\right]\;. \tag{3.48}$$

We thus recover the structures discussed in the previous subsection, cf. (3.34)–(3.36).

4 Conclusions

As discussed, e.g., in [4, 10, 11], Euclidean supersymmetry differs in many ways from the usual Lorentzian one. In this article this is again illustrated by considering the target space geometry of a "natural" sigma model in Euclidean signature. We encountered the modification of the complex geometry defined by the complex tensor J which formally satisfies the usual definitions of complex structure, but is now a complex tensor. We considered in detail the geometry that emerges from off-shell supersymmetry which differers from the usual case both in the signature of the metric and in the additional structure it carries. The geometry is described by $(M, g, B, f, \tilde{f})$ where f and $\tilde{f}$ are Yano f-structures when the superymmetry algebra closes off-shell. This structure is derived from a potential as in the Lorentzian case. The field equations are well-defined provided only that H is globally defined, but the quantum theory requires further that H represent a trivial cohomology class. It is only the special case of a Kähler manifold as target space that can be described by $N = 2$-supersymmetric models of all three kinds discussed here.

Acknowledgments

The research of UL was supported by EU grant (Superstring theory) MRTN-2004-512194 and VR grant 621- 2006-3365. The research of R.v.U. was supported by the Simons Center for Geometry and Physics as well as the Czech ministry of education under contract No. MSM0021622409. The research of M.Z. was supported by VR-grant 621-2008-4273. All the authors are happy to thank the program "Geometrical Aspects of String Theory" at Nordita, where part of this work was carried out.

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